Inverting the Hamitonian Reduction in String Theory[1]

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It is well known that many interesting realisations of string theories can be obtained via hamiltonian reduction from WZW models. I want to point out that string theories do in certain cases also provide the recipe to reconstruct the ambient space of the hamiltonian reduction, including Kač–Moody currents and the associated ghosts. The procedure of reconstructing the Kač–Moody currents is closely related to properties of matter+gravity multiplets in noncritical string theories. In application to KPZ gravity and its $N=1$ supersymmetric extension, the 'inverted hamiltonian reduction' constructions serve to establish relation with the DDK-type formalism for matter + gravity.

Introduction

The importance of embedding of non-critical strings into WZW models was pointed out recently in [2]. Such embeddings would in a number of cases look like inverting the quantum Hamiltonian reduction [3, 4, 5, 6]. Generally, 'inverting the Hamiltonian reduction' is an ill-posed problem, as is the inversion of any projection:

$$
\begin{array}{ccc}
K & \longrightarrow & G \\
& & \downarrow \pi \\
& & G/K
\end{array}
\tag{1}
$$

Constructing π^{-1} involves non-canonical choices of representatives etc. However, in conformal field theory an 'almost canonical' choice for these non-canonical data happens to be provided by coupling to (super)gravity. Matter + gravity theories possess an '$N \mapsto N + 2$ supersymmetry enhancement' [7, 5]. While not every $N + 2$-supersymmetric model is necessarily an N-matter dressed with gravity, *demanding* it to be so provides one with almost all the data necessary to invert the hamiltonian reduction. The only remaining piece is then provided by tensoring the theory with a Fock module that is a 'bosonisation' of the matter Verma module. In this way the ambient space of the reduction, i.e. the original Kač–Moody algebra *and* the 'Drinfeld–Sokolov' ghosts that are necessary to build up the BRST complex, can be reconstructed, at least for those cases when the hamiltonian reduction results in a linear algebra.

As a 'practical' application of the 'inverted Hamiltonian reduction' I will consider, following [1], the KPZ formulation [8, 9, 10, 11, 12, 13, 14, 15] of the induced (super)gravity. While the above-mentioned 'supersymmetry enhancement' can be considered as a fundamental property of gravity in the conformal gauge [26, 27], another fundamental property of two-dimensional gravity is the existence, in the light-cone gauge, of an $s\ell(2)$ Kač–Moody symmetry or its super-extension [14, 15]. While the $s\ell(2)$ current algebra, as it was derived, appears completely independent of the $N = 2$ formalism, one may nevertheless wonder what is the relation between conformal field-theoretic descriptions of the two formalisms for two-dimensional gravity. They cannot be literally embedded into one another, as the underlying symmetry algebras would not allow that.

Recently, a more general concept of relations between conformal field theories has emerged in the context of Universal String Theory [17, 18, 19, 20, 21, 22, 23, 24]. In this approach, one realises theories with lower symmetries as 'special backgrounds' of those with higher symmetries. This construction actually considers conformal field theories modulo trivial topological theories: the extra fields that 'decouple' do in fact form a trivial topological sector, which does not bring in any new physical states. In the spirit of Universal String Theory applied to the *gravity* sector, I will 'evaluate' the KPZ theory on a background provided by a representation for the current algebra obtained by the 'inverted Hamiltonian reduction'. The result turns out to be the DDK formulation of matter+gravity models, plus an extra topological piece. Thus the DDK theory arises as a 'special background' of the KPZ theory. Conversely, when the (super)DDK theory is tensored with a certain topological field theory, a hidden (super-)$s\ell(2)$ symmetry emerges. It will be discussed below in which sense the 'extra' topological theory can be considered *trivial*; this involves a fermionic screening operator similar to the one discussed in [5, 6, 25].

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[1] Talk given at the 28th International Symposium on the Theory of Elementary Particles, Wendisch-Rietz, August 30 – September 3, 1994

This talk is based on a work with Chris Hull [1].

Inverting the hamiltonian reduction

• The **topological conformal algebra** [29, 30] is generated by an energy-momentum tensor $\mathcal{T}$, a bosonic dimension-one current $\mathcal{H}$, a fermionic dimension-one (BRST) current $\mathcal{J}$ and a fermionic dimension-two current $\mathcal{G}$. These generators are realised in any conformal field theory made up by a matter theory with energy-momentum tensor T of central charge d, bc ghosts that have dimensions $1, 0$ [7], and a single scalar ϕ with background charge α_0 chosen so that the total central charge vanishes, $d - 2 + (1 + 6\alpha_0^2) = 0$.

The generators of the topological conformal algebra read

$$
\begin{aligned}
\mathcal{T} &= T - \tfrac{1}{2}(\partial\phi)^2 + \frac{\alpha_0}{\sqrt{2}}\partial^2\phi - b\partial c\,, \qquad \mathcal{J} = b\,, \\
\mathcal{G} &= c\Big(T - \tfrac{1}{2}(\partial\phi)^2 + \frac{\alpha_0}{\sqrt{2}}\partial^2\phi\Big) - b\partial c{\cdot}c + \sqrt{2}\alpha_+\partial c{\cdot}\partial\phi + \tfrac{1}{2}(1 - 2\alpha_+^2)\partial^2 c\,, \\
\mathcal{H} &= -bc - \sqrt{2}\alpha_+\partial\phi
\end{aligned}
\tag{2}
$$

The parameters $\alpha_{\pm,0}$ satisfy $\alpha_0 = \alpha_+ + \alpha_-$, $\alpha_+\alpha_- = -1$ as usual, and it will be useful to introduce a level k by $\alpha_+ = \frac{1}{\sqrt{k+2}}$, $\alpha_- = -\sqrt{k+2}$.

• I am going to show that by adding an extra scalar v_* to the system of fields from the RHSs or (2), it is possible to **construct an $sl(2)_k$ algebra**, and in fact a (twisted) $sl(2)_k \oplus u(1)_{BC}$ algebra (where $u(1)_{BC}$ is actually a bosonisation of a pair of fermionic ghosts).

Among the building blocks there are, therefore, an energy-momentum tensor T with central charge $d = 1 - 6\alpha_0^2$, a 'Liouville' scalar ϕ with background charge $\alpha_0/\sqrt{2}$, a pair of ghosts b, c of dimensions $1, 0$, and the v_* scalar. Then a twisted $N{=}2$ conformal algebra is generated by (2), while the $sl(2)_k$ currents are constructed as

$$
\begin{aligned}
J^+ &= e^{\sqrt{2}\alpha_+(v_* - \phi)}\,, \qquad J^0 = bc + \sqrt{2}\alpha_+\partial\phi - \Big(\frac{\alpha_-}{\sqrt{2}} + \sqrt{2}\alpha_+\Big)\partial v_*\,, \\
J^- &= \Big\{\alpha_-^2\Big(T - \tfrac{1}{2}(\partial\phi)^2 + \frac{\alpha_0}{\sqrt{2}}\partial^2\phi\Big) + 2b\partial c - \alpha_-^2\partial(bc) + \sqrt{2}\alpha_-\partial\phi{\cdot}bc\Big\}e^{-\sqrt{2}\alpha_+(v_* - \phi)}
\end{aligned}
\tag{3}
$$

Moreover, the 'Drinfeld–Sokolov' BC ghosts (which are a fermionisation of the $u(1)$ current), in turn, can be constructed as

$$
B = ce^{\sqrt{2}\alpha_+(v_* - \phi)}\,, \qquad C = be^{-\sqrt{2}\alpha_+(v_* - \phi)}\,.
\tag{4}
$$

The formulae (3)–(4) invert the (quantum) Drinfeld–Sokolov reduction, in the sense that, given an abstract matter theory that represents the result of the reduction ($J^- \sim T$), eqs. (3), (4) show how to add 'dressing' fields that would lead to reconstructing the ambient space of the hamiltonian reduction, together with the ghost system that is required in order to build up the Drinfeld–Sokolov BRST complex.

Evaluating the twisted Sugawara energy-momentum tensor

$$
\widetilde{T^{\rm S}} = \frac{1}{k+2}\big(J^0 J^0 + \tfrac{1}{2}(J^+ J^- + J^- J^+)\big) + \partial J^0\,.
\tag{5}
$$

with the currents being given by (3), one arrives at the identity

$$
\widetilde{T^{\rm S}} + \partial B{\cdot}C = \mathcal{T} + \tfrac{1}{2}(\partial v_*)^2 - \frac{\alpha_0}{\sqrt{2}}\partial^2 v_*
\tag{6}
$$

where $\mathcal{T}$ is the energy-momentum tensor from (2). Eq. (6) thus represents two ways to describe the same theory: $sl(2) \oplus u(1)_{BC} \simeq \text{(topological)} \oplus (\partial v_*)$.

Moreover, admissible $sl(2)$ representations can be arrived at starting from topological algebra representations tensored with the v_*-matter sector. BRST-invariant primary states of the topological conformal algebra are characterised by their topological $U(1)$ charge:

$$
\mathcal{H}_0|h\rangle_{\rm top} = h|h\rangle_{\rm top}\,, \qquad \mathcal{L}_{\geq 0}|h\rangle_{\rm top} = \mathcal{H}_{\geq 1}|h\rangle_{\rm top} = \mathcal{G}_{\geq 1}|h\rangle_{\rm top} = \mathcal{J}_{\geq 0}|h\rangle_{\rm top} = 0\,.
\tag{7}
$$

In the representation (2) of the topological algebra, such states are constructed from matter dressed with ghosts and Liouville, and are given by

$$\big|\mathsf{h}(r,s)\big\rangle_{\text{top}} = \big|r,s\big\rangle \otimes \big|p_{\mathrm{M}}(r,s)\big\rangle_{\mathrm{L}} \otimes \big|0\big\rangle_{bc}, \tag{8}$$

where $p_{\mathrm{M}}(r,s) = -\frac{1}{\sqrt{2}}(\alpha_+(r-1) + \alpha_-(s-1))$, and $\big|p\big\rangle_{\mathrm{L}}$ corresponds to $e^{p\phi}$ in the Liouville sector. The topological $U(1)$ charge is evaluated as $\mathsf{h}(r,s) = -\alpha_+^2(r-1) + s - 1$. Admissible $s\ell(2)_k$ highest-weight primary states of spin

$$j(r,s) = \frac{r-1}{2} - (k+2)\frac{s-1}{2} \tag{9}$$

are arrived at (tensored with the ghost vacuum $\big|0\big\rangle \equiv \big|0\big\rangle_{BC}$) as follows:

$$\big|j(r,s)\big\rangle_{s\ell(2)} \otimes \big|0\big\rangle_{BC} = \big|r,s\big\rangle \otimes \big|p_{\mathrm{M}}(r,s)\big\rangle_{\mathrm{L}} \otimes \big|0\big\rangle_{bc} \otimes \big|-p_{\mathrm{M}}(r,s)\big\rangle_* \tag{10}$$

On the LHS of (10), we have thus obtained a primary state of the algebra $s\ell(2)_k \oplus u(1)_{BC}$.

The RHS of (10) can also be read as a dressing of the matter operator $U_{r,s} \sim \big|r,s\big\rangle$ with

$$V_{r,s} = e^{-p_{\mathrm{M}}(r,s)(v_* - \phi)} = e^{j(r,s)\sqrt{\frac{2}{k+2}}(v_* - \phi)}. \tag{11}$$

• To extend the previous constructions to $N=1$ **supergravity**, I start with an $N=1$ matter that comprises an energy-momentum tensor T_{m} and its superpartner G_{m}:

$$
\begin{aligned}
T_{\mathrm{m}}(z)T_{\mathrm{m}}(w) &= \frac{d/2}{(z-w)^4} + \frac{2T_{\mathrm{m}}(w)}{(z-w)^2} + \frac{\partial T_{\mathrm{m}}(w)}{z-w}, \\
T_{\mathrm{m}}(z)G_{\mathrm{m}}(w) &= \frac{3/2\,G_{\mathrm{m}}(w)}{(z-w)^2} + \frac{\partial G_{\mathrm{m}}(w)}{z-w}, \qquad G_{\mathrm{m}}(z)G_{\mathrm{m}}(w) = \frac{2d/3}{(z-w)^3} + \frac{2T_{\mathrm{m}}(w)}{z-w},
\end{aligned}
\tag{12}
$$

where $d = \frac{15}{2} - 3\alpha_-^2 - 3\alpha_+^2$ is the matter cental charge and $\alpha_- = -1/\alpha_+$. When coupling this system to supergravity, one introduces a super-Liouville field, with components ϕ, ψ, and fermionic and bosonic ghosts bc and $\beta\gamma$. The energy-momentum tensor is given by

$$\mathcal{T} = T_{\mathrm{m}} - \tfrac{1}{2}\partial\phi\,\partial\phi + \tfrac{1}{2}(\alpha_+ - \alpha_-)\partial^2\phi - \tfrac{1}{2}\partial\psi\,\psi - \partial b\,c - 2b\partial c - \tfrac{3}{2}\beta\partial\gamma - \tfrac{1}{2}\partial\beta\,\gamma. \tag{13}$$

The full theory admits an $N=3$ symmetry algebra whose generators can be constructed as explained in [5]. First, the three supersymmetry generators are given by

$$
\begin{aligned}
\mathcal{G}^+ &= b, \qquad \mathcal{G}^0 = -G_{\mathrm{m}} + b\gamma - 2c\partial\beta - 3\partial c\beta - \psi\partial\phi + (\alpha_+ - \alpha_-)\,\partial\psi, \\
\mathcal{G}^- &= 4cT_{\mathrm{m}} + 2\gamma G_{\mathrm{m}} - b\gamma\gamma - 4b\partial c\,c - 2c\beta\partial\gamma - 2c\partial\phi\,\partial\phi + 2c\partial\beta\,\gamma - 2c\partial\psi\,\psi + (-2\alpha_- + 2\alpha_+ - 8x)c\partial^2\phi \\
&\quad + 2\psi\gamma\partial\phi + 4x\psi\partial\gamma - 8x\partial c\,\partial\phi + 4\partial c\,\beta\gamma + (2\alpha_- - 2\alpha_+ + 4x)\partial\psi\,\gamma - 8x^2\partial^2 c
\end{aligned}
\tag{14}
$$

where two values of the parameter x are possible, $x = \frac{1}{2}\alpha_+$ or $x = -\frac{1}{2}\alpha_-$. Next, there exists a Majorana–Weyl fermion $\mathcal{F} = -2x\psi + c\beta$. Operator products of the supersymmetry generators with $\mathcal{F}$ generate three bosonic currents, $\mathcal{G}^a(z)\mathcal{F}(w) = \frac{\mathcal{H}^a}{z-w}$, where

$$
\begin{aligned}
\mathcal{H}^+ &= \beta, \qquad \mathcal{H}^0 = -2x\partial\phi + bc + \beta\gamma, \\
\mathcal{H}^- &= -2bc\gamma - \beta\gamma\gamma - 2c\psi\partial\phi + (-2\alpha_- + 2\alpha_+ - 8x)c\partial\psi + 4x\gamma\partial\phi + 2G_{\mathrm{m}}c - 4x\partial c\,\psi + 2\partial c\,c\beta + 8x^2\partial\gamma.
\end{aligned}
\tag{15}
$$

These satisfy an $s\ell(2)$ algebra at level $-8x^2$:

$$\mathcal{H}^0(z)\mathcal{H}^\pm(w) = \pm\frac{\mathcal{H}^\pm}{z-w}, \quad \mathcal{H}^0(z)\mathcal{H}^0(w) = \frac{-4x^2}{(z-w)^2}, \quad \mathcal{H}^+(z)\mathcal{H}^-(w) = \frac{-8x^2}{(z-w)^2} + \frac{2\mathcal{H}^0}{z-w} \tag{16}$$

Now, before completing the list of commutation relations among the $N=3$ generators, observe that the (twisted, to be precise) $N=3$ algebra admits an involutive automorphism that consists in $\mathcal{G}^\pm \mapsto \mathcal{G}^\mp$, $\mathcal{H}^\pm \mapsto \mathcal{H}^\mp$, $\mathcal{H}^0 \mapsto -\mathcal{H}^0$, and twisting $\mathcal{T} \mapsto \mathcal{T} + \partial\mathcal{H}^0$, $\mathcal{G}^0 \mapsto -\mathcal{G}^0 - 2\partial\mathcal{F}$. As I will need only the second, transformed, version, I will not distinguishing

between the two constructions related by the automorphism, and will simply consider the above constructions for $\mathcal{G}^\pm$ and $\mathcal{H}^{\pm,0}$ as belonging to the algebra that contains the twisted energy-momentum tensor

$$\mathcal{T} = T_{\mathrm{m}} - \tfrac{1}{2}\partial\phi\partial\phi + \tfrac{1}{2}(\alpha_+ - \alpha_- - 4x)\partial^2\phi - \tfrac{1}{2}\partial\psi\,\psi - b\partial c - \tfrac{1}{2}\beta\partial\gamma + \tfrac{1}{2}\partial\beta\,\gamma \tag{17}$$

and the $\mathcal{G}^0$ generator

$$\mathcal{G}^0 = -G_{\mathrm{m}} + b\gamma - \partial c\beta - \psi\partial\phi + (\alpha_+ - \alpha_- - 4x)\partial\psi \tag{18}$$

(the previous $\mathcal{T}$ and $\mathcal{G}^0$ will not appear any more). I also choose the first option for the parameter from (14) $x = \tfrac{1}{2}\alpha_+$. It can be checked then that, in addition to (16), the following relations hold:

$$
\begin{aligned}
\mathcal{H}^+(z)\mathcal{G}^0(w) &= \frac{-\mathcal{G}^+}{z-w}, & \mathcal{H}^+(z)\mathcal{G}^-(w) &= \frac{2\mathcal{F}(w)}{(z-w)^2} + \frac{2\mathcal{G}^0 - 2\partial\mathcal{F}}{z-w}, \\
\mathcal{H}^0(z)\mathcal{G}^\pm(w) &= \frac{\pm\mathcal{G}^\pm}{z-w}, & \mathcal{H}^0(z)\mathcal{G}^0(w) &= \frac{\mathcal{F}(w)}{(z-w)^2}, \\
\mathcal{H}^-(z)\mathcal{G}^+(w) &= \frac{2\mathcal{F}(w)}{(z-w)^2} + \frac{-2\mathcal{G}^0 + 2\partial\mathcal{F}}{z-w}, & \mathcal{H}^-(z)\mathcal{G}^0(w) &= \frac{\mathcal{G}^-}{z-w}.
\end{aligned}
\tag{19}
$$

and

$$
\begin{aligned}
\mathcal{G}^0(z)\mathcal{G}^+(w) &= \frac{\mathcal{H}^+(w)}{(z-w)^2} + \frac{\partial\mathcal{H}^+}{z-w}, & \mathcal{G}^+(z)\mathcal{G}^-(w) &= \frac{-16x^2}{(z-w)^3} + \frac{4\mathcal{H}^0(w)}{(z-w)^2} + \frac{4\mathcal{T}}{z-w}, \\
\mathcal{G}^0(z)\mathcal{G}^-(w) &= \frac{-3\mathcal{H}^-(w)}{(z-w)^2} + \frac{-\partial\mathcal{H}^-}{z-w}, & \mathcal{G}^0(z)\mathcal{G}^0(w) &= \frac{2\mathcal{T}}{z-w}.
\end{aligned}
\tag{20}
$$

The energy-momentum tensor (17) assigns $(\mathcal{G}^+, \mathcal{G}^0, \mathcal{G}^-; \mathcal{H}^+, \mathcal{H}^0, \mathcal{H}^-)$ dimensions $(1, \tfrac{3}{2}, 2; \tfrac{1}{2}, 1, \tfrac{3}{2})$.

• Now I proceed to the **construction of an** $osp(1|2)$ **algebra** and the associated 'Drinfeld–Sokolov' ghosts. Just as in the purely bosonic case, let us add to the system of supermatter coupled to $N=1$ supergravity a matter theory explicitly represented by a scalar field ∂v_* with the energy-momentum tensor

$$T_* = \tfrac{1}{2}\partial v_*\partial v_* + \tfrac{1}{2}(\alpha_+ + \alpha_-)\partial^2 v_* \tag{21}$$

Then, the $osp(1|2)$ currents are constructed as

$$
\begin{aligned}
J^+ &= e^{2\alpha_+(\phi - v_*)}, \quad j^+ = \sqrt{2}\psi e^{\alpha_+(\phi - v_*)}, \quad J^0 = \tfrac{1}{2}\alpha_-\partial\phi + bc + \tfrac{1}{2}\alpha_-\sqrt{7\alpha_+^2 - 2\beta\gamma}, \\
j^- &= \Big(\alpha_-\sqrt{\frac{1+\alpha_+^2}{2}}\psi\partial\Phi + \sqrt{2}bc\psi \mp \frac{\alpha_-}{\sqrt{2}}G_{\mathrm{m}} + \frac{\alpha_-^2 - 2}{\sqrt{2}}\partial\psi\Big)e^{-\alpha_+(\phi - v_*)} \\
J^- &= \frac{1}{2\alpha_+^2}\Big(-\tfrac{1}{2}(1+\alpha_+^2)\partial\Phi\partial\Phi + \tfrac{1}{2}(3\alpha_+ - \alpha_-)\sqrt{1+\alpha_+^2}\,\partial^2\Phi + 2\alpha_+\sqrt{1+\alpha_+^2}\,bc\partial\Phi \\
&\quad - (1 - 3\alpha_+^2)b\partial c - (1 + \alpha_+^2)\partial b\,c + (1 + \alpha_+^2)T_{\mathrm{eff}}\Big)e^{-2\alpha_+(\phi - v_*)}
\end{aligned}
\tag{22}
$$

where useful combinations were introduced as

$$
\begin{aligned}
\partial\Phi &= \frac{\alpha_+}{\sqrt{1+\alpha_+^2}}\Big(-(2\alpha_- + 3\alpha_+)\partial\phi + (\alpha_- + 3\alpha_+)\partial v_* - \alpha_-\sqrt{7\alpha_+^2 - 2\beta\gamma}\Big) \\
T_{\mathrm{eff}} &= \frac{1}{1+\alpha_+^2}T_{\mathrm{m}} \pm \frac{\alpha_+}{1+\alpha_+^2}G_{\mathrm{m}}\psi + \frac{1 - 2\alpha_+^2}{2(1+\alpha_+^2)}\partial\psi\,\psi.
\end{aligned}
\tag{23}
$$

One must be careful to first evaluate in (22) the normal-ordered product $:\!\partial\Phi\partial\Phi\!:$ and then, $::\!\partial\Phi\partial\Phi\!:e^{-2\alpha_+(\phi - v_*)}:$. It can be checked that the currents (22) satisfy the $osp(1|2)$ algebra:

$$
\begin{aligned}
J^+(z)J^-(w) &= \frac{k}{(z-w)^2} + \frac{2J^0}{z-w}, & J^0(z)J^0(w) &= \frac{k/2}{(z-w)^2}, \\
j^+(z)j^-(w) &= \frac{-2k}{(z-w)^2} - 2\frac{J^0}{z-w}, & J^0(z)J^\pm(w) &= \pm\frac{J^\pm}{(z-w)}, \\
J^0(z)j^\pm(w) &= \pm\tfrac{1}{2}\frac{j^\pm}{z-w}, & J^\pm(z)j^\mp(w) &= -\frac{j^\pm}{z-w}, \quad j^\pm(z)j^\pm(w) = \mp 2\frac{J^\pm}{z-w}
\end{aligned}
\tag{24}
$$

where now, in the supersymmetric setting, the level k is introduced by $k = (\alpha_-^2 - 3)/2$.

Next, the fermionic (BC) and bosonic $(\boldsymbol{\beta\gamma})$ ghosts are given by

$$
\begin{aligned}
B &= b e^{-2\alpha_+(\phi - v_*)}, & C &= c e^{2\alpha_+(\phi - v_*)}, \\
\boldsymbol{\beta} &= \beta e^{\sqrt{7\alpha_+^2 - 2}(\phi - v_*)}, & \boldsymbol{\gamma} &= \gamma e^{-\sqrt{7\alpha_+^2 - 2}(\phi - v_*)}
\end{aligned}
\tag{25}
$$

• Now let us consider the ghosts from (25), along with the $osp(1|2)$ currents, as independent fields. Note that, according to their explicit constructions, $(B, C, \boldsymbol{\beta}, \boldsymbol{\gamma})$ are given dimensions $(1, 0, \frac{1}{2}, \frac{1}{2})$ by the energy-momentum tensor $\mathcal{T} + T_*$. Therefore, the appropriate energy-momentum tensor reads

$$
T_{\text{ghosts}} = -B\partial C + \tfrac{1}{2}\partial\boldsymbol{\beta}\,\boldsymbol{\gamma} - \tfrac{1}{2}\boldsymbol{\beta}\partial\boldsymbol{\gamma}
\tag{26}
$$

Then, T_{ghosts} and the Sugawara energy-momentum tensor

$$
T_{\text{Sug}} = 2\alpha_+^2 \left(J^0 J^0 + \tfrac{1}{2}J^+ J^- + \tfrac{1}{2}J^- J^+ + \tfrac{1}{4}j^+ j^- - \tfrac{1}{4}j^- j^+ \right) + \partial J^0
\tag{27}
$$

describe the full system in terms of the 'composite' fields $(B, C, \boldsymbol{\beta}, \boldsymbol{\gamma}, J^+, j^+, J^0, j^-, J^-)$. Note that T_{Sug} assigns the currents $(J^+, j^+, J^0, j^-, J^-)$ dimensions $(0, \frac{1}{2}, 1, \frac{3}{2}, 2)$ respectively. Evaluating $T_{\text{ghosts}} + T_{\text{Sug}}$ in terms of the 'elementary' fields $(T_{\text{m}}, G_{\text{m}}, b, c, \beta, \gamma, \partial\phi, \psi, \partial v_*)$, one finds

$$
T_{\text{Sug}} + T_{\text{ghosts}} = \mathcal{T} + T_*
\tag{28}
$$

This is some sort of the 'completeness relation', showing that no degrees of freedom are lost when trading supermatter coupled to supergravity and the additional T_* piece for an $osp(1|2)$ algebra with the corresponding ghosts. This fact does not seem obvious a priori, in particular considering that the matter theory is not necessarily bosonised through a free superfield (nor are the $osp(1|2)$ currents free).

Applications to the KPZ Formulation

The right-moving sector of the gauge-fixed theory is a conformal field theory consisting of the matter system, the $s\ell(2)_k$ current algebra corresponding to the gravitational degrees of freedom, and two ghost systems corresponding to the two gauge conditions [8, 9, 10, 11, 12, 13]. Choose the matter sector to be a minimal model with energy-momentum tensor, T', satisfying the Virasoro algebra with central charge $d' = 1 - 6\alpha_0'^2$ and define $\alpha_+'\alpha_-' = -1$, $\alpha_+' + \alpha_-' = \alpha_0'$.

Primary states $\left| r', s' \right\rangle'$ of the matter sector are labelled by two integers r' and s' and have dimensions

$$
\Delta'(r', s') = \tfrac{1}{4}\Big(\alpha_+'(r' - 1) + \alpha_-'(s' - 1) \Big)\Big(\alpha_+'(r' + 1) + \alpha_-'(s' + 1) \Big)
\tag{29}
$$

In the $sl(2)_k$ sector, a highest-weight spin-j state $\left| j \right\rangle$ has twisted Sugawara dimension

$$
\widetilde{\Delta}^{\text{S}}(j) = \frac{j(j + 1)}{k + 2} - j.
\tag{30}
$$

Admissible $s\ell(2)_k$ representations are built upon highest-weight states $\left| j \right\rangle$ whose spin is given by $j = j_1 - j_2(k + 2)$ for some (half-) integers j_1 and j_2.

There are also two bc ghost systems, $b^{[2]}c^{[2]}$ with spins $2, -1$ and $b^{[0]}c^{[0]}$ with dimensions $0, 1$ respectively. The total energy-momentum tensor is then

$$
T^{\text{KPZ}} = T' + \widetilde{T^{\text{S}}} + t^{[2]} + t^{[0]}.
\tag{31}
$$

The BRST current resulting from the gauge-fixing is

$$
\mathcal{J}_{\text{BRST}}^{\text{KPZ}}(z) = c^{[2]}\Big(T' + \widetilde{T^{\text{S}}} + \tfrac{1}{2}t^{[2]} + t^{[0]} \Big) + c^{[0]}J^+ \equiv \mathcal{J}^{[2]}(z) + \mathcal{J}^{[0]}(z).
\tag{32}
$$

For the BRST charge Q^{KPZ} to be nilpotent, the total central charge must vanish, which determines $k_\pm = -2 - \alpha_\pm'^2$.

Now, BRST-invariant states can be sought in the form

$$
\left| r', s' \right\rangle' \otimes \left| j \right\rangle_{s\ell(2)} \otimes c_1^{[2]}\left| 0 \right\rangle^{[2]} \otimes \left| 0 \right\rangle^{[0]}.
\tag{33}
$$

Vanishing of Q^{KPZ} on such a state determines j in terms of r' and s'. There are two solutions related by $\alpha'_\pm \leftrightarrow \alpha'_\mp$, $r' \leftrightarrow s'$, and the one I choose to work with reads

$$k = -2 - {\alpha'_-}^2 \qquad j(r',s') = \begin{cases} \frac{r'-1}{2} - (k+2)\frac{-s'-1}{2} \\[2mm] \frac{-r'-1}{2} - (k+2)\frac{s'-1}{2} \end{cases} \tag{34}$$

• Consider now the KPZ theory in which the $s\ell(2) \oplus u(1)_{b^{[0]}c^{[0]}}$ generators are realised in terms of the topological ingredients (a matter system T dressed with a scalar ϕ and dimension-one ghosts b, c) and the ∂v_* scalar, The resulting theory is then precisely the DDK theory of matter dressed with Liouville gravity, plus ghosts, tensored with a ('new') topological field theory. The latter is trivial when reduced by the action of appropriate screening charges, so that one obtains an explicit construction of the DDK theory from the KPZ theory. This works as follows.

The **energy-momentum tensor** of the $s\ell(2)$ and dimension-zero ghost system $\widetilde{T^{\mathrm{S}}} + t^{[0]}$ is given by (6), with the identification $B \equiv b^{[0]}$, $C \equiv c^{[0]}$. Substituting this into the KPZ energy-momentum tensor given by (31) and using the construction (2) for the topological energy-momentum tensor, eq. (31) becomes

$$T^{\mathrm{KPZ}} = T' + T - \partial b^{[2]} c^{[2]} - 2b^{[2]}\partial c^{[2]} - \tfrac{1}{2}(\partial\phi)^2 + \frac{\alpha_0}{\sqrt{2}}\partial^2\phi - b\partial c + T_* , \tag{35}$$

where, as in (6),

$$T_* = \tfrac{1}{2}(\partial v_*)^2 - \frac{\alpha_0}{\sqrt{2}}\partial^2 v_* . \tag{36}$$

Now notice that, by virtue of (34), $d' = 13 + 6/(k+2) + 6(k+2)$, whence $d + d' = 26$ and therefore the first four terms in (35) are a good candidate to describe the DDK-type theory. It actually remains to compare the expression for the $s\ell(2)$ spin j, which one gets from the KPZ theory, with the expression for the same thing that follows from (10), (9). It follows that *either* $(r' = -r,\ s' = s)$ *or* $(r' = r,\ s' = -s)$, and therefore the dimensions in the matter and in the matter$'$ sectors (i.e. those with energy-momentum tensors T, T') add up to 1:

$$\Delta'(r',s') + \Delta(r,s) = 1 , \tag{37}$$

which is the DDK prescription. The two 'matter' theories thus play the dual rôles of a 'true matter' and a 'Liouville'.

Further, the central charge for the $\partial\phi$-∂v_*-bc sector vanishes and this sector is in fact topological by itself. This is because its energy-momentum tensor is of the form of the one from (2), but with its matter part T replaced by T_*. This $*$-matter has the same central charge $d = 1 - 6\alpha_0^2$, which fits the Liouville central charge and the ghosts' dimensions to make up a topological algebra according to the construction (2). The *only* modification in the formulae for the new topological generators is the replacement in (2) of T with T_*.

It is the topological theory thus obtained that represents the 'difference' between the KPZ and DDK formulations. Equivalence of the KPZ and DDK descriptions requires that the 'extra' $*$-topological theory be empty. However, in the way it has emerged in (35), this theory seems to possess all the non-trivial states given by a specialisation of the previous construction, namely by eq. (8) in which $|r,s\rangle$ is now replaced by the free-field realisation. That is, the $*$-topological primary states are given by

$$\big|-p_{\mathrm{M}}(r,s)\big\rangle_* \otimes \big|p_{\mathrm{M}}(r,s)\big\rangle_{\mathrm{L}} \otimes |0\rangle_{bc} , \tag{38}$$

in which one recognises (11)!

Recall, however, that when using free-field constructions, the price to be paid for apparent simplifications is the necessity of introducing screening (and/or picture-changing) operators. For example, the Wakimoto bosonisation of $s\ell(2)$ provides three screening operators, of which two are bosonic and one fermionic. This would also have been the case with our representation (3) for the $s\ell(2)$ curents, had the matter sector been bosonised through a scalar, via $T \to \tfrac{1}{2}(\partial u)^2 - \frac{\alpha_0}{\sqrt{2}}\partial^2 u$. Then, by a field redefinition one would be able to map formulae (3) into a more standard Wakimoto form [33, 34, 35], and the two standard bosonic screening operators from the Wakimoto representation would then be mapped back into $e^{-\sqrt{2}\alpha_+ u}$ and $e^{-\sqrt{2}\alpha_- u}$. This shows that the screenings in our representation (3) belong entirely to the matter sector and do not involve the other fields ϕ, v_*, and bc from (3).

In general, T in eqs. (3) can be the energy-momentum tensor of an arbitrary matter system. Then, unless T is specifically chosen to be realised in terms of a Coulomb gas representation, the corresponding screenings are not relevant to the representation (3) of the $s\ell(2)_k$ algebra. However, there does exist a sector in the theory that *is* bosonised through a scalar, and therefore requires screening operators. This is the $*$-topological algebra given by the same construction (2)

in which one replaces T with T_*, the energy-momentum tensor for the scalar v_*. The corresponding fermionic screening current (cf. [5]) has the form:

$$\mathcal{S}_* = b e^{\alpha_-(v_* - \phi)/\sqrt{2}} \tag{39}$$

This is completely 'OPE-isotropic', i.e. $\mathcal{S}_*(z)\mathcal{S}_*(w) = 0$, and in addition satisfies the OPE's

$$\mathcal{J}(z)\mathcal{S}_*(w) = 0\,, \qquad \mathcal{H}(z)\mathcal{S}_*(w) = 0\,. \tag{40}$$

The nilpotent operator $\mathcal{Q}^{[*]} = \oint \mathcal{S}_*$ can be used as a BRST charge. There are, therefore, two BRST charges, $\mathcal{Q}$ and $\mathcal{Q}^{[*]}$ and one can define different theories by demanding that the physical states be in the cohomology of $\mathcal{Q}$ or of $\mathcal{Q}^{[*]}$, or of the linear combination $Q^{\text{t.m.m.}} = \mathcal{Q} + \mathcal{Q}^{[*]}$, or one can demand the physical states to be in the double complex, i.e. to be simultaneously in the cohomology of both $\mathcal{Q}$ and $\mathcal{Q}^{[*]}$. For example, whether or not a topological theory constructed by dressing matter with ghosts and Liouville is equivalent to a topological minimal model depends on the definition of the BRST operator, i.e. on whether or not $\mathcal{Q}^{[*]}$ is added to the BRST charge [5].

Consider the case in which physical states are simultaneously in the cohomology of both $\mathcal{Q}$ and $\mathcal{Q}^{[*]}$, described in [5] as a 'naive' case. This gives a much greater physical state space reduction. This makes sense in the current setup, because the $*$-topological sector is only a part of a larger theory, and, according to (40), the action of $\mathcal{Q}^{[*]}$ would preserve the $\mathcal{Q}$-BRST invariance as well as topological $U(1)$ charges of topological primary fields. Moreover, with respect to the 'full' energy-momentum tensor (6), $\mathcal{S}_*$ is a screening current as well.

The $*$-topological algebra shares the BRST current $\mathcal{J} \equiv b$ with the topological algebra (2). A simple similarity transformation on the generators,

$$\mathcal{X}_* \mapsto \left(\exp \frac{\alpha_-}{\sqrt{2}} \oint bc(v_* - \phi)\right) \mathcal{X}_* \left(\exp -\frac{\alpha_-}{\sqrt{2}} \oint bc(v_* - \phi)\right) \tag{41}$$

maps between the two BRST currents, $\mathcal{J} \mapsto \mathcal{S}_*$, and changes the other generators accordingly. In this way, $\mathcal{S}_*$ becomes a BRST current of another twisted $N = 2$ algebra. All the states (38), corresponding to operators (11), remain unchanged under this transformation, and therefore the algebra resulting from (41) would, too, act on these states. However, all these states are obviously BRST-trivial in this algebra, i.e. with respect to $\mathcal{Q}^{[*]} = \oint \mathcal{S}_*$:

$$V_{r,s} = \{\mathcal{Q}^{[*]}, cV_{r,s-1}\} \tag{42}$$

Therefore, in the $*$-sector, corresponding to the 'coset' KPZ/DDK, (i.e., roughly, KPZ = DDK $\oplus$ $*$-topological), the states (38) are eliminated from the space of physical states by 'strengthening' the procedure of [5], namely by choosing the double cohomology of $\mathcal{Q}^{[*]}$ and $\mathcal{Q}$. Note that the $\mathcal{Q}^{[*]}$ operator can obviously be lifted up to the 'full' theory described by energy-momentum tensor $\mathbb{T}$ (it is essential that $[\mathbb{T}(z), \mathcal{Q}^{[*]}] = 0$). In this sense it can be thought of as having been present in the theory from the start, and the corresponding reduction of the space of physical states makes a part of the definition of the theory. With this definition adopted, one does indeed get a special gravitational background of the KPZ theory on which it reduces to the DDK theory tensored with a *trivial* topological theory (consisting only of ground states). Once the operator $\mathcal{Q}^{[*]}$ is 'lifted up' to the whole KPZ theory, the operation performed in (42) can be written as $\{\mathcal{Q}^{[*]}, b^{[0]}(J^+)^{-1}(\cdot)\}$, where the current $\mathcal{S}_*$ itself becomes, formally,

$$\mathcal{S}_* = c^{[0]}(J^+)^{-k/2} \tag{43}$$

which might be interpreted along the lines of ref. [36].

• Now consider the $osp(1|2)$-**supergravity**. In the matter sector, one has an $N = 1$-supersymmetric matter″-theory described by T_m'' and G_m'' satisfying relations (12) in which d is replaced by $d'' = \frac{15}{2} + 3\alpha_-^2 + 3\alpha_+^2$. Along with this matter system one introduces [14] a free Majorana–Weyl fermion χ with the energy-momentum tensor $T_\chi = \frac{1}{2}\partial\chi\chi$, whence central charge $\frac{1}{2}$. Coupling matter″ to supergravity can be described then by an $osp(1|2)$ algebra and a set of bosonic and fermionic ghosts: $b^{[2]}c^{[2]}$ as the usual reparametrisation ghosts, $\beta^{[\frac{3}{2}]}\gamma^{[\frac{3}{2}]}$ as the super-ghosts, and also dimension-1 ghosts $b^{[1]}c^{[1]}$ and their superpartners $\beta^{[\frac{1}{2}]}\gamma^{[\frac{1}{2}]}$, corresponding to the gauging of J^+ and j^+ currents respectively.

Now, combine the $osp(1|2)$ algebra with the $b^{[1]}c^{[1]}$ and $\beta^{[\frac{1}{2}]}\gamma^{[\frac{1}{2}]}$ ghosts (to be identified with BC and $\boldsymbol{\beta\gamma}$ respectively) and express it in terms of the topological ingredients and the v_*-matter. One will thus have the following field content and the corresponding central charges:

matter″	χ	matter	bc	$\beta\gamma$	$\partial\phi$	ψ	∂v_*	$b^{[2]}c^{[2]}$	$\beta^{[\frac{3}{2}]}\gamma^{[\frac{3}{2}]}$
$\frac{15}{2} + 3\alpha_+^2 + 3\alpha_-^2$	$\frac{1}{2}$	$\frac{15}{2} - 3\alpha_+^2 - 3\alpha_-^2$	-2	-1	$-5 + 3\alpha_+^2 + 3\alpha_-^2$	$\frac{1}{2}$	$7 - 3\alpha_+^2 - 3\alpha_-^2$	-26	11

$$(44)$$

Here, $d'' + d = 15$, and the $b^{[2]}c^{[2]}$ and $\beta^{[\frac{3}{2}]}\gamma^{[\frac{3}{2}]}$ ghosts contribute $-26 + 11 = -15$. One can therefore combine matter$''$ + matter + $b^{[2]}c^{[2]}$ + $\beta^{[\frac{3}{2}]}\gamma^{[\frac{3}{2}]}$ into a super-Distler–Kawai sector, in which the matter and matter$''$ theories play the dual rôles of 'proper' matter and Liouville (recall that each of the two matter sectors is $N\!=\!1$-supersymmetric).

On top of that, another central charge-0 theory comprises

$$
\begin{array}{cccccc}
bc & \partial\phi & \partial v_* & \chi & \beta\gamma & \psi \\
-2 & (-5 + 3\alpha_+^2 + 3\alpha_-^2) & (7 - 3\alpha_+^2 - 3\alpha_-^2) & \frac{1}{2} & -1 & \frac{1}{2}
\end{array}
\tag{45}
$$

Here, ∂v_* and χ can be combined into a supersymmetric $*$-matter, which is then dressed with the other fields into a twisted $N\!=\!3$ algebra as above, the difference from (14) and (15) being that the $T_{\rm m}$ and $G_{\rm m}$ generators are now given in terms of the free fields:

$$
T_{{\rm m}*} = \tfrac{1}{2}\partial v_*\partial v_* + \tfrac{1}{2}(\alpha_+ + \alpha_-)\partial^2 v_* + \tfrac{1}{2}\partial\chi\,\chi, \qquad G_{{\rm m}*} = \partial v_*\,\chi + (\alpha_+ + \alpha_-)\partial\chi
\tag{46}
$$

A characteristic difference from the bosonic construction is the fact that the ∂v_* theory is supersymmetrised by a fermion, χ, that comes from the $osp(1|2)$ gravity. This 'corrects' the fact that in the super-matter sector, the fermionic part brings in a $\frac{1}{2}$, whereas ∂v_* is only a free boson, with a $\frac{1}{2}$ thus missing from the central charge. The ∂v_* field does actually get its superpartner from the 'extra ghost' of the $osp(1|2)$ gravity. Reversing the argument, i.e. constructing the $osp(1|2)$ currents and the associated ghosts from the super-Distler–Kawai fields and the $*$-$N\!=\!3$ algebra, we thus see that this extra ghost is nothing but a survived superpartner to the ∂v_* matter.

We have seen, among other things, that, just as a large class of string theories have a twisted $N\!=\!2$ superconformal algebra, a certain class of conformal field theories have a hidden $sl(2)$ Kač–Moody algebra.

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